# On the solvability of some inhomogeneous incompressible flow with free interface (Mathematical Analysis in Fluid and Gas Dynamics)



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# On the solvability of some inhomogeneous incompressible flow with free interface

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**Abstract**

In this note, we review some recent results in [4] on the solvability of some free boundary problem of the two phase inhomogeneous incompressible Navier-Stokes equations. In [4], we addressed some general approach to construct short time solutions in $L_p - L_q$ maximal regularity class where $q > N$ and $p$ belongs to $]2, \infty[ \bigcup \{\bar{p} \in ]1, 2[ : 1/\bar{p} + N/q > 3/2\}$. In particular, to handle the less regular initial data for $1 < p < 2$, some new estimates are derived. Moreover, for the case of piecewise constant density, some long time solutions in the moving bounded droplet are establish within $L_p - L_q$ maximal regularity. Furthermore, we can find some global solutions in the fixed bounded pool by applying the idea in [4] here.

## 1 Introduction

### 1.1 Model

Consider the motion of two immisible fluids in the bulks $\dot{\Omega}_t := \Omega_{+,t} \cup \Omega_{-,t} \subset \mathbb{R}^N$ with $N \geq 2$, divided by some free sharp interface $\Gamma_t \neq \emptyset$. In general, we suppose that $\partial\,\Omega_{+,t} = \Gamma_{+,t} \cup \Gamma_t$ and $\partial\,\Omega_{-,t} = \Gamma_t \cup \Gamma_-$ for some free surface $\Gamma_{+,t}$ and some fixed hypersurface $\Gamma_-$. Moreover, $\boldsymbol{n}_t$ and $\boldsymbol{n}_{+,t}$ are outward unit normals subject to $\Gamma_t$ and $\Gamma_{+,t}$ respectively at time $t$. With such settings on $\dot{\Omega}_t$, we shall study the following Cauchy problem without taking the surface tension into account,

$$\begin{cases} \partial_t(\rho\boldsymbol{v}) + \mathrm{Div}(\rho\boldsymbol{v}\otimes\boldsymbol{v}) - \mathrm{Div}\,\mathbb{T}(\boldsymbol{v},\mathfrak{p}) = \rho\boldsymbol{f} & \text{in} \quad \dot{\Omega}_t, \\ \partial_t\rho + \mathrm{div}(\rho\boldsymbol{v}) = 0, \ \ \mathrm{div}\,\boldsymbol{v} = 0 & \text{in} \quad \dot{\Omega}_t, \\ [\![\mathbb{T}(\boldsymbol{v},\mathfrak{p})\boldsymbol{n}_t]\!] = \boldsymbol{0}, \ \ [\![\boldsymbol{v}]\!] = \boldsymbol{0}, \ \ V_t = \boldsymbol{v}\cdot\boldsymbol{n}_t & \text{on} \quad \Gamma_t, \\ \mathbb{T}(\boldsymbol{v}_+,\mathfrak{p}_+)\boldsymbol{n}_{+,t} = \boldsymbol{0}, \ \ V_{+,t} = \boldsymbol{v}_+\cdot\boldsymbol{n}_{+,t} & \text{on} \quad \Gamma_{+,t}, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on} \quad \Gamma_-, \\ (\rho,\boldsymbol{v})|_{t=0} = (\rho_0,\boldsymbol{v}_0) & \text{on} \quad \dot{\Omega}. \end{cases} \tag{$INS_\pm$}$$

In $(INS_\pm)$, our aim is to determine the unknowns $(\rho, \boldsymbol{v}, \mathfrak{p}, \dot{\Omega}_t)$: the density, the velocity field, the pressure and the moving domain, whenever the external force $\boldsymbol{f}$ and initial states $(\rho_0, \boldsymbol{v}_0, \dot{\Omega})$ are given. In addition, the viscous stress tensor $\mathbb{T}(\boldsymbol{v}, \mathfrak{q})$ is defined by [1]

$$\mathbb{T}(\boldsymbol{v},\mathfrak{q}) := \mu(\rho)\mathbb{D}(\boldsymbol{v}) - \mathfrak{q}\mathbb{I} \ \text{ with } \ \mathbb{D}(\boldsymbol{v}) := \nabla^\top\boldsymbol{v} + \nabla\boldsymbol{v}^\top,$$

[1] $\nabla_\xi^\top \boldsymbol{Y}$ stands for the Jacobian matrix of $\boldsymbol{Y}$, i.e. $(\nabla_\xi^\top \boldsymbol{Y})_k^j := \partial_{\xi_k} Y^j$ with $1 \leq j, k \leq N$, and $\nabla_\xi \boldsymbol{Y}^\top := (\nabla_\xi^\top \boldsymbol{Y})^\top$.

where $\mu$ is some smooth strictly positive function. In addition, $V_t$ and $V_{+,t}$ stand for the normal velocity of moving surfaces $\Gamma_t$ and $\Gamma_{+,t}$ respectively. The jump of the vector $\boldsymbol{g}$ across some surface $\mathcal{S}$ is given by the following non-tangential limit

$$[\![\boldsymbol{g}]\!](x_0) := \lim_{\delta\to 0+} \Big(\boldsymbol{g}\big(x_0 + \delta\boldsymbol{\nu}(x_0)\big) - \boldsymbol{g}\big(x_0 - \delta\boldsymbol{\nu}(x_0)\big)\Big) \quad \forall\ x_0 \in \mathcal{S},$$

where $\boldsymbol{\nu}$ is the unit outwards normal along the surface $\mathcal{S}$. For the history of the free boundary value problems of the viscous flows, we refer to [4, Sec.1]. In the following, we mainly focus on the mathematical results.

### 1.2 Reduction of $(INS_\pm)$ in Lagrangian coordinates

Motivated by the work [7] due to V.A.Solonnikov, we take advantage of the so-called *Lagrangian coordinates* to study $(INS_\pm)$,

$$\boldsymbol{X}_u(\xi,t) := \xi + \int_0^t \boldsymbol{u}(\xi,\tau)\,d\tau \quad \text{for all } \xi \in \dot{\Omega} = \Omega_+ \cup \Omega_-. \tag{1.1}$$

In fact, $\boldsymbol{X}_u$ stands for the trajectory of $\boldsymbol{v}$, that is $\boldsymbol{u}(\xi,t) := \boldsymbol{v}\big(\boldsymbol{X}_u(\xi,t),t\big)$. Moreover, we have $(\Gamma_t, \Gamma_{+,t}, \Gamma_-) = \boldsymbol{X}_u\big((\Gamma,\Gamma_+,\Gamma_-),t\big)$ for the boundaries $\Gamma$, $\Gamma_+$ and $\Gamma_-$ of $\dot{\Omega}$. By this means, $(INS_\pm)$ is reduced to some problem on the fixed domain $\dot{\Omega}$. To write down the new equations under (1.1), we adopt the following conventions.

- $\mathscr{A}_u$ stands for the cofactor matrix of $\nabla_\xi^\top \boldsymbol{X}_u$. Moreover $\nabla_u := \mathscr{A}_u\nabla_\xi$, $\operatorname{div}_u = \operatorname{Div}_u := \nabla_u \cdot$.
- Note that $\rho\big(\boldsymbol{X}_u(\xi,t),t\big) = \rho_0(\xi)$ and set $\mathfrak{q}(\xi,t) := \mathfrak{p}\big(\boldsymbol{X}_u(\xi,t),t\big)$, then the corresponding stress tensor
$$\mathbb{T}_u(\boldsymbol{u},\mathfrak{q}) := \mu(\rho_0)\mathbb{D}_u(\boldsymbol{u}) - \mathfrak{q}\mathbb{I} \ \text{ with } \ \mathbb{D}_u(\boldsymbol{u}) := \nabla_\xi^\top \boldsymbol{u}\cdot\mathscr{A}_u^\top + \mathscr{A}_u\cdot\nabla_\xi \boldsymbol{u}^\top.$$
- Suppose that $\boldsymbol{n}$ and $\boldsymbol{n}_+$ are the unit normal for $\Gamma$ and $\Gamma_+$ respectively. Define that
$$(\overline{\boldsymbol{n}}, \overline{\boldsymbol{n}}_+)(\xi,t) := (\boldsymbol{n}_t, \boldsymbol{n}_{+,t})\big(\boldsymbol{X}_u(\xi,t)\big) = \Big(\frac{\mathscr{A}_u\boldsymbol{n}}{|\mathscr{A}_u\boldsymbol{n}|}, \frac{\mathscr{A}_{u_+}\boldsymbol{n}_+}{|\mathscr{A}_{u_+}\boldsymbol{n}_+|}\Big)(\xi,t), \quad \forall \xi\in\Gamma\cup\Gamma_+.$$

Thanks to $(INS_\pm)$, it is not hard to verify that $(\boldsymbol{u},\mathfrak{q})$ satisfies

$$\begin{cases} \rho_0\partial_t\boldsymbol{u} - \operatorname{Div}_u \mathbb{T}_u(\boldsymbol{u},\mathfrak{q}) = \rho_0\boldsymbol{f}\big(\boldsymbol{X}_u(\xi,t),t\big),\ \operatorname{div}_u \boldsymbol{u} = 0 & \text{in} \quad \dot{\Omega}\times]0,T[,\\ [\![\mathbb{T}_u(\boldsymbol{u},\mathfrak{q})\overline{\boldsymbol{n}}]\!] = [\![\boldsymbol{u}]\!] = \boldsymbol{0} & \text{on} \quad \Gamma\times]0,T[,\\ \mathbb{T}_{u_+}(\boldsymbol{u}_+,\mathfrak{q}_+)\overline{\boldsymbol{n}}_+ = \boldsymbol{0} & \text{on} \quad \Gamma_+\times]0,T[,\\ \boldsymbol{u}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times]0,T[,\\ \boldsymbol{u}|_{t=0} = \boldsymbol{v}_0 & \text{on} \quad \dot{\Omega}. \end{cases} \tag{$INS_\pm^{\mathfrak{L}}$}$$

In the rest of this note, we will attack the wellposedness issues concerning $(INS^{\mathfrak{L}}_{\pm})$ instead of $(INS_{\pm})$, because the solvability of $(INS_{\pm})$ can be reduced to the study of $(INS^{\mathfrak{L}}_{\pm})$ in our framework via some standard arguments.

## 2 Main results

### 2.1 Domains and viscosity coefficient

To reveal the results of $(INS^{\mathfrak{L}}_{\pm})$, let us first specify the assumptions on $\dot{\Omega}$ and $\mu$.

**Definition.** *We say that a connected open subset $\Omega$ in $\mathbb{R}^N$ $(N \geq 2)$ is of class $W^{2-1/r}_r$ for some $1 < r < \infty$, if and only if for any point $x_0 \in \partial\Omega$, one can choose a Cartesian coordinate system with origin $x_0$ (up to some translation and rotation) and coordinates $y = (y', y_N) := (y_1, ..., y_{N-1}, y_N)$, as well as positive constants $\alpha, \beta, K$ and some $W^{2-1/r}_r$ function $h$ satisfying $\|h\|_{W^{2-1/r}_r} \leq K$ such that the neighborhood of $x_0$*

$$U_{\alpha,\beta,h}(x_0) := \{(y', y_N) : h(y') - \beta < y_N < h(y') + \beta, |y'| < \alpha\}$$

*satisfies*

$$U^-_{\alpha,\beta,h}(x_0) := \{(y', y_N) : h(y') - \beta < y_N < h(y'), |y'| < \alpha\} = \Omega \cap U_{\alpha,\beta,h}(x_0),$$

*and*

$$\partial\Omega \cap U_{\alpha,\beta,h}(x_0) = \{(y', y_N) : y_N = h(y'), |y'| < \alpha\}.$$

*Above $\alpha, \beta, K, h$ may vary with respect to the different location on the boundary. Whenever the choices of $\alpha, \beta, K$ are independent of the position of $x_0$, $\Omega$ is called uniform $W^{2-1/r}_r$ domain. Note that if the boundary $\partial\Omega$ is compact, then the uniformness is satisfied automatically. Sometimes $\Omega$ is just called $W^{2-1/r}_r$ regular for simplicity.*

Now we admit the following assumptions in this context.

($\mathcal{H}1$) $\dot{\Omega}$ is uniformly $W^{2-1/r}_r$ for some $r > N$, i.e. $\Omega_{\pm}$ are uniformly $W^{2-1/r}_r$ domains;

($\mathcal{H}2$) $\mu\big(\rho_0(x)\big)$ is a strictly positive function on $\dot{\Omega}$ satisfying

$$\underline{\mu}_+ \mathbb{1}_{\Omega_+} + \underline{\mu}_- \mathbb{1}_{\Omega_-} \leq \mu\big(\rho_0(\cdot)\big) \leq \bar{\mu}_+ \mathbb{1}_{\Omega_+} + \bar{\mu}_- \mathbb{1}_{\Omega_-},$$

where $\underline{\mu}_{\pm}$ and $\bar{\mu}_{\pm}$ are all strictly positive constants. In addition, we assume that $\mu \in C^1(\mathbb{R}_+; \mathbb{R}_+)$.

## 2.2 Some weak elliptic problem and the reduced Stokes operator

Set $\Omega := \Omega_+ \cup \Omega_- \cup \Gamma$ for $\dot\Omega$ in $(\mathcal{H}1)$, and let us introduce several useful functional spaces and the Stokes operator for two phase problem. The standard Sobolev space is denoted by $W^m_q(\Omega)$ for any $m \in \mathbb{N}$ and $q \in ]1,\infty[$, while $\widehat{W}^1_q(\Omega)$ stands for the homogeneous space, i.e.

$$\widehat{W}^1_q(\Omega) := \{f \in L_{q,loc}(\Omega) : \|f\|_{\widehat{W}^1_q(\Omega)} := \|\nabla f\|_{L_q(\Omega)} < \infty\}.$$

Next, the linear space $X^1_{q,\Gamma_+}(\Omega)$ for any $1 < q < \infty$ is defined as below,

$$X^1_{q,\Gamma_+}(\Omega) := \begin{cases} \{f \in X^1_q(\Omega) : f = 0 \text{ on } \Gamma_+\} & \text{if } \Gamma_+ \neq \emptyset, \\ X^1_q(\Omega) & \text{if } \Gamma_+ = \emptyset, \end{cases}$$

with the word $X \in \{W, \widehat{W}\}$ and $\|f\|_{X^1_{q,\Gamma_+}(\Omega)} := \|f\|_{X^1_q(\Omega)}$. For any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ defined in some domain $G \subset \mathbb{R}^N$, denote that

$$(\boldsymbol{u}, \boldsymbol{v})_G := \int_G \boldsymbol{u} \cdot \boldsymbol{v}\, dx = \sum_{j=1}^N \int_G u^j v^j\, dx.$$

Now recall the so-called *weak elliptic transmission problem.*

**Definition.** *Consider some domain $\Omega$ as above. Suppose that $1 < q < \infty$ and the step function $\eta := \eta_+ \mathbb{1}_{\Omega_+} + \eta_- \mathbb{1}_{\Omega_-}$ for some constants $\eta_\pm > 0$. Then we say that the weak elliptic transmission problem is uniquely solvable on $\widehat{W}^1_{q,\Gamma_+}(\Omega)$ for $\eta$ if the following assertions hold true: For any $\boldsymbol{f} \in L_q(\Omega)^N$, there is a unique $\theta \in \widehat{W}^1_{q,\Gamma_+}(\Omega)$ (up to some constant) satisfying,*

$$(\eta^{-1}\nabla\theta, \nabla\varphi)_{\dot\Omega} = (\boldsymbol{f}, \nabla\varphi)_{\dot\Omega} \quad \textit{for all } \varphi \in \widehat{W}^1_{q',\Gamma_+}(\Omega).$$

*Moreover, there exists a constant $C$ independent on the choices of $\theta$, $\varphi$ and $\boldsymbol{f}$ such that*

$$\|\nabla\theta\|_{L_q(\Omega)} \leq C\|\boldsymbol{f}\|_{L_q(\Omega)}.$$

With the definition above, one more hypothesis for our domain $\Omega$ is added as below,

($\mathcal{H}3$) The weak elliptic transmission problem is uniquely solvable on $\widehat{W}^1_{q,\Gamma_+}(\Omega)$ and $\widehat{W}^1_{q',\Gamma_+}(\Omega)$ for some $\eta_\pm > 0$ and some $1 < q < \infty$.

**Remark.** *Let us make some comments on the assumption* $(\mathcal{H}3)$.

1. *The choice of $\widehat{W}^1_{q,\Gamma_+}(\Omega)$ is more general than the definition in [2] since our approach is also expected for the domain with some exterior bulk. Moreover, according to $(\mathcal{H}3)$, we may introduce the hydrodynamic Lebesgue space*

$$J_q(\dot\Omega) := \{\boldsymbol{f} \in L_q(\Omega)^N : (\boldsymbol{f}, \nabla\varphi)_{\dot\Omega} = 0,\ \forall\varphi \in \widehat{W}^1_{q',\Gamma_+}(\Omega)\}.$$

2. *Now, define the functional space* $\boldsymbol{W}_q^{-1}(\Omega)$ $(1<q<\infty)$ *by*

$$\boldsymbol{W}_q^{-1}(\Omega) := \{g \in L_q(\Omega) : \exists\, \boldsymbol{R} \in L_q(\Omega)^N \text{ such that } (g,\varphi)_\Omega = -(\boldsymbol{R}, \nabla\varphi)_\Omega,\ \ \forall\ \varphi \in W^1_{q',\Gamma_+}(\Omega)\}$$

*which will be useful later. Here let us point that the definition of* $\boldsymbol{W}_q^{-1}(\Omega)$ *makes sense. For instance, we will see* $\boldsymbol{W}_q^{-1}(\Omega) \neq \emptyset$ *if* $\Gamma_+ \neq \emptyset$. *To this end, let us denote the dual of any Banach space* $E$ *by* $E^\star$, *namely,* $E^\star := \mathcal{L}(E;\mathbb{R})$. *Then we introduce that*

$$\mathcal{W}_q^{-1}(\Omega) := \begin{cases} \big(\widehat{W}^1_{q',\Gamma_+}(\Omega)\big)^\star & \text{if } \Gamma_+ \neq \emptyset, \\ \big(\dot{W}^1_{q'}(\Omega)\big)^\star & \text{if } \Gamma_+ = \emptyset, \end{cases}$$

*with* $\dot{W}^1_q(\Omega) := \{\, [\theta]_1 : \theta \in \widehat{W}^1_q(\Omega)\}$ *and* $[\theta]_1 := \{\theta + c : c \in \mathbb{R}\}$. *Here* $\langle\cdot,\cdot\rangle_\Omega$ *stands for the corresponding pair due to the definition of* $\mathcal{W}_q^{-1}(\Omega)$. *Moreover, set*

$$\widetilde{L}_q(\Omega) := L_q(\Omega)^N / J_q(\dot{\Omega}) = \{\, [\boldsymbol{G}]_2 : \boldsymbol{G} \in L_q(\Omega)^N\}$$

*and* $[\boldsymbol{G}]_2 := \{\boldsymbol{G} + \boldsymbol{f} : \boldsymbol{f} \in J_q(\Omega)\}$. *Then by adapting the arguments in [2], there exists* $\mathcal{G}(g) := [\boldsymbol{G}]_2 \in \widetilde{L}_q(\Omega)$ *for any* $g \in \mathcal{W}_q^{-1}(\Omega)$ *such that*

$$\langle g, [\varphi]\rangle_\Omega = -(\boldsymbol{G}, \nabla\varphi)_\Omega \ \text{ for any } \ \varphi \in \mathcal{W}^1_{q'}(\Omega). \tag{2.1}$$

$[\varphi]$ *above stands for* $[\varphi]_1$ *if* $\Gamma_+ = \emptyset$ *and* $[\varphi] = \varphi$ *otherwise. In particular,* (2.1) *yields that*

$$(g,\varphi)_\Omega = -(\mathfrak{g}, \nabla\varphi)_\Omega \ \text{ for any } \ (\mathfrak{g},\varphi) \in \mathcal{G}(g) \times W^1_{q',\Gamma_+}(\Omega),$$

*provided that* $g \in L_q(\Omega) \cap \mathcal{W}_q^{-1}(\Omega)$. *Thus we can conclude* $L_q(\Omega) \cap \mathcal{W}_q^{-1}(\Omega) \subset \boldsymbol{W}_q^{-1}(\Omega)$ *for the case* $\Gamma_+ \neq \emptyset$.

3. *As another consequence of* $(\mathcal{H}3)$, *if we set for any* $\boldsymbol{u} \in W^2_q(\dot{\Omega})^N$ $(1<q<\infty)$,

$$\begin{aligned} \boldsymbol{\alpha}_u &:= \eta^{-1} \operatorname{Div}\big(\mu\mathbb{D}(\boldsymbol{u})\big) - \nabla \operatorname{div} \boldsymbol{u}, \\ \beta_u &:= [\![\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}]\!]\boldsymbol{n} - [\![\operatorname{div} \boldsymbol{u}]\!], \\ \gamma_u &:= \big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}_+\big)\boldsymbol{n}_+ - \operatorname{div} \boldsymbol{u}, \end{aligned}$$

*then there exists a unique mapping* $K(\boldsymbol{u}) := \theta \in W^1_q(\dot{\Omega}) + \widehat{W}^1_{q,\Gamma_+}(\Omega)$ *satisfying*

$$(\eta^{-1}\nabla\theta, \nabla\varphi)_{\dot{\Omega}} = (\boldsymbol{\alpha}_u, \nabla\varphi)_{\dot{\Omega}}, \quad [\![\theta]\!] = \beta_u \ \text{ on } \Gamma \quad \text{and} \quad \theta = \gamma_u \ \text{ on } \Gamma_+.$$

Next, given $\eta := \eta_+\mathbb{1}_{\Omega_+} + \eta_-\mathbb{1}_{\Omega_-}$ for $\eta_\pm > 0$, $\mathcal{A}_q\boldsymbol{u} := \eta^{-1}\operatorname{Div}\mathbb{T}\big(\boldsymbol{u}, K(\boldsymbol{u})\big)$ is exactly the (reduced) Stokes operator for two phase problem with its domain

$$\mathcal{D}(\mathcal{A}_q) := \{\boldsymbol{u} \in W^2_q(\dot{\Omega})^N \cap J_q(\dot{\Omega}) :\ [\![\boldsymbol{u}]\!]|_\Gamma = [\![\mathcal{T}_{\boldsymbol{n}}\big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}\big)]\!]|_\Gamma = \boldsymbol{0}, \ \mathcal{T}_{\boldsymbol{n}_+}\big(\mu\mathbb{D}(\boldsymbol{u})\boldsymbol{n}_+\big)|_{\Gamma_+} = \boldsymbol{0},\ \ \boldsymbol{u}|_{\Gamma_-} = \boldsymbol{0}\}.$$

$\mathcal{T}_{\boldsymbol{\nu}}\boldsymbol{h} := \boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{\nu})\boldsymbol{\nu}$ above is a projection into the hypersurface orthogonal to $\boldsymbol{\nu}$ for any vector $\boldsymbol{\nu}$ and $\boldsymbol{h}$ defined along some surface $\mathcal{S}$. Then our short time result for $(INS^{\mathfrak{c}}_{\pm})$ reads as below.

**Theorem 2.1.** *Let $(p,q)$ be in the sets $(I)\cup(II)$ with*

$$(I) := \{(p,q) \in ]2,\infty[\times]N,\infty[\} \quad \textit{and} \quad (II) := \{(p,q) \in ]1,2]\times]N,\infty[ : 1/p + N/q > 3/2\}.$$

*Additionally, hypotheses $(\mathcal{H}1)$-$(\mathcal{H}3)$ are fulfilled and $\eta$ is given as above. Assume that $\rho_0 \in \widehat{W}^1_q(\dot\Omega)$, $\boldsymbol{v}_0$ is in $\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega) := \big(J_q(\dot\Omega), \mathcal{D}(\mathcal{A}_q)\big)_{1-1/p,p}$ and $\boldsymbol{f}$ belongs to $L_p\big(0,2;W^1_\infty(\mathbb{R}^N)^N\big)$. If, in addition,$\|\eta-\rho_0\|_{L_\infty(\dot\Omega)} \leq c$ for some constant $c \ll 1$, then there are some constants $T(<1)$ and $C$, only depending on $p$, $q$, $\boldsymbol{v}_0$ and $\boldsymbol{f}$, such that $(INS^{\mathfrak{c}}_{\pm})$ admits a unique solution $(\boldsymbol{u},\mathfrak{q})$ satisfying*

$$\|\boldsymbol{u}\|_{L_p(0,T;W^2_q(\dot\Omega))\cap W^1_p(0,T;L_q(\dot\Omega))} + \|\nabla\mathfrak{q}\|_{L_p(0,T;L_q(\dot\Omega))} \leq C.$$

*In addition, if $\mu$ is piecewise constant, we can relax the constrain $\rho_0 \in \widehat{W}^1_q(\dot\Omega)$ to $\rho_0 \in L_\infty(\dot\Omega)$.*

Inspired by results for the one phase flow in [6], case $(I)$ above in somehow easier as the embedding $\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega) \hookrightarrow W^1_q(\dot\Omega)$. However, our discussion of case $(II)$ is based on more refined interpolation arguments, whose explanation is postponed to the comments after the global-in-time result.

### 2.3 Some long time solution in the case of bounded droplet

Now, let $\Omega := \Omega_+ \cup \Omega_- \cup \Gamma$ be some bounded droplet satisfying $(\mathcal{H}1)$ with $\Gamma_- = \emptyset$. Moreover, the hypothesis $(\mathcal{H}3)$ is fulfilled for any $\eta := \eta_+\mathbb{1}_{\Omega_+} + \eta_-\mathbb{1}_{\Omega_-}$ $(\eta_\pm > 0)$ due to [5] by Y.Shibata. Our second result in [4] is about the unique long time solution of $(INS^{\mathfrak{c}}_{\pm})$ for such domain with piecewise constant density. To this end, let us introduce the rigid motion space

$$\mathcal{R}_d := \{\boldsymbol{p}(x) = \mathbb{A}x + \boldsymbol{b} : \ \mathbb{A}\,\text{is an}\,N\times N\,\text{anti-symmetric matrix and}\ \boldsymbol{b}\in\mathbb{R}^N\}.$$

Without loss of generality, set $M := \dim\mathcal{R}_d \in \mathbb{N}$ and then there exist a basis family

$$\mathfrak{P} := \{\boldsymbol{p}_\alpha \in \mathcal{R}_d : (\eta\,\boldsymbol{p}_\alpha, \boldsymbol{p}_\beta)_{\dot\Omega} = \delta^\alpha_\beta, \ \text{ for any } 1\leq\alpha,\beta\leq M\},$$

such that $\mathcal{R}_d := span\{\,\boldsymbol{p}_\alpha \in \mathfrak{P}\}$. Now some long time solutions in $L_p - L_q$ maximal regularity class can be established as follows.

**Theorem 2.2.** *Let $(p,q)\in(I)\cup(II)$ as in Theorem 2.1 and $\Omega$ be a bounded $W^{2-1/r}_r$ $(r\geq q)$ droplet with $\Gamma_- = \emptyset$. Assume that $\rho_0(\xi) = \eta = \eta_+\mathbb{1}_{\Omega_+} + \eta_-\mathbb{1}_{\Omega_-}$ and $\mu = \mu_+\mathbb{1}_{\Omega_+} + \mu_-\mathbb{1}_{\Omega_-}$*

*are piecewise constants for any* $\eta_\pm, \mu_\pm > 0$. *If* $\|\boldsymbol{v}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)} \ll 1$ *such that* $(\eta \boldsymbol{v}_0, \boldsymbol{p}_\alpha)_{\dot\Omega} = 0$ *for any* $\boldsymbol{p}_\alpha \in \mathfrak{P}$, *then* $(INS^{\mathfrak{L}}_\pm)$ *admits a unique global solution* $(\boldsymbol{u}, \mathfrak{q})$. *Moreover, there exists constant* $\varepsilon_0$ *and* $C$ *such that*

$$\|e^{\varepsilon_0 t}\boldsymbol{u}\|_{W^{2,1}_{q,p}(\dot\Omega\times]0,T[)} + \|e^{\varepsilon_0 t}\mathfrak{q}\|_{L_p(0,T;W^1_q(\dot\Omega))} \leq C\|\boldsymbol{v}_0\|_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)} \text{ for any } T > 0.$$

Let us make some comments on the index set $(II)$ in Theorem 2.1 and Theorem 2.2. In fact, the motivation of the set $(II)$ is due to the following product law.

**Lemma 2.3.** *Let* $(\theta, \alpha, \beta, q, p) \in ]0,1[\times[q,\infty]^2\times]N,\infty[\times[1,2]$ *satisfy*

$$\frac{1}{q} = \frac{1}{\alpha} + \frac{1}{\beta}, \quad 1 - \frac{\theta}{p} = \frac{N}{q} - \frac{N}{\alpha} \quad \text{and} \quad 1 - \frac{2(1-\theta)}{p} = \frac{N}{q} - \frac{N}{\beta}.$$

*Assume that* $g \in H^{1/2,1/2}_{q,p}(G\times\mathbb{R})$ *and* $f \in L_\infty(\mathbb{R}; W^1_q(G))$ *fulfilling* $\partial_t f \in L_{p/\theta}\big(\mathbb{R}; L_\beta(G)\big)$. *Then there exists a constant* $C_{p,q}$ *such that*

$$\|fg\|_{H^{1/2,1/2}_{q,p}(G\times\mathbb{R})} \leq C_{p,q}\|f\|^{1/2}_{L_\infty(G\times\mathbb{R})}\Big(\|f\|_{L_\infty(\mathbb{R};W^1_q(G))} + \|\partial_t f\|_{L_{p/\theta}(\mathbb{R};L_\beta(G))}\Big)^{1/2}\|g\|_{H^{1/2,1/2}_{q,p}(G\times\mathbb{R})}.$$

Thanks to the constrain in Lemma 2.3, we have

$$N/q + 1/p = 3/2 + N/(2\alpha) > 3/2,$$

which gives our definition of $(II)$ in the main results.

Another fundamental tool to obtain Theorem 2.2 is the decay property of two phase Stokes system. The natural linearized procedure of $(INS^{\mathfrak{L}}_\pm)$ reads,

$$\begin{cases} \partial_t \boldsymbol{u} - \eta^{-1}\operatorname{Div}\mathbb{T}(\boldsymbol{u},\mathfrak{q}) = \boldsymbol{f}, \ \operatorname{div}\boldsymbol{u} = g = \operatorname{div}\boldsymbol{R} & \text{in} \quad \dot\Omega\times\mathbb{R}_+, \\ [\![\mathbb{T}(\boldsymbol{u},\mathfrak{q})\boldsymbol{n}]\!] = [\![\boldsymbol{h}]\!], \ [\![\boldsymbol{u}]\!] = \boldsymbol{0} & \text{on} \quad \Gamma\times\mathbb{R}_+, \\ \mathbb{T}_+(\boldsymbol{u}_+,\mathfrak{q}_+)\boldsymbol{n}_+ = \boldsymbol{k} & \text{on} \quad \Gamma_+\times\mathbb{R}_+, \\ \boldsymbol{u}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times\mathbb{R}_+, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{u}_0 & \text{in} \quad \dot\Omega. \end{cases} \tag{2.2}$$

However, only piecewise constant viscosity case is taken into account for simplicity. Namely,

$$\mu := \mu_+\mathbb{1}_{\Omega_+} + \mu_-\mathbb{1}_{\Omega_-} \text{ for some constants } \mu_\pm > 0. \tag{2.3}$$

Furthermore, we introduce several functional spaces for convenience to shorten our description.

- Recall the rigid motion space $\mathcal{R}_d$ and its basis $\mathfrak{P}$ used in Theorem 2.2. Then we adopt that

$$\widetilde{\mathcal{D}}^{2-2/p}_{q,p}(\dot\Omega) := \begin{cases} \{\boldsymbol{u} \in \mathcal{D}^{2-2/p}_{q,p}(\dot\Omega) : (\eta\boldsymbol{u}, \boldsymbol{p}_\alpha)_{\dot\Omega} = 0, \ \forall\, \boldsymbol{p}_\alpha \in \mathfrak{P}\} & (\Gamma_- = \emptyset), \\ \mathcal{D}^{2-2/p}_{q,p}(\dot\Omega) & (\Gamma_- \neq \emptyset). \end{cases}$$

Moreover, we would like to take the constant $\delta(\Gamma_-) = 1$ for $\Gamma_- = \emptyset$ and otherwise set $\delta(\Gamma_-) = 0$.

- In addition, we say $(\boldsymbol{f}, g, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{k}) \in \mathcal{Z}_{p,q,\varepsilon_0}$ for some $1 < p, q < \infty$ and $\varepsilon_0 > 0$, if $\boldsymbol{f}, g, \boldsymbol{R}, \boldsymbol{h}$ and $\boldsymbol{k}$ satisfy the conditions,

$$e^{\varepsilon_0 t}\boldsymbol{f} \in L_{p,0}\big(\mathbb{R}; L_q(\dot\Omega)\big)^N,\ e^{\varepsilon_0 t} g \in H^{1,1/2}_{q,p,0}(\dot\Omega\times\mathbb{R}) \cap L_{p,0}\big(\mathbb{R}; \boldsymbol{W}^{-1}_q(\Omega)\big),$$
$$e^{\varepsilon_0 t}(\partial_t \boldsymbol{R}, \boldsymbol{R}) \in L_{p,0}\big(\mathbb{R}; L_q(\dot\Omega)\big)^{2N},\ e^{\varepsilon_0 t}\boldsymbol{h} \in H^{1,1/2}_{q,p,0}(\dot\Omega\times\mathbb{R})^N \text{ and } e^{\varepsilon_0 t}\boldsymbol{k} \in H^{1,1/2}_{q,p,0}(\Omega_+\times\mathbb{R})^N.$$

Moreover, the norm $\|\cdot\|_{\mathcal{Z}_{p,q,\varepsilon_0}}$ is given by

$$\|(\boldsymbol{f}, g, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{k})\|_{\mathcal{Z}_{p,q,\varepsilon_0}} := \|e^{\varepsilon_0 t}(\boldsymbol{f}, \boldsymbol{R}, \partial_t\boldsymbol{R})\|_{L_p(\mathbb{R}_+; L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g, \boldsymbol{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbb{R})}$$
$$+ \|e^{\varepsilon_0 t}\boldsymbol{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbb{R})}.$$

With above symbols, we summarize the decay properties of (2.2) proved in [3, 4].

**Theorem 2.4.** *Assume that $1 < p, q < \infty$, $N < r < \infty$ and $r \geq \max\{q, q/(q-1)\}$. Suppose that $\Omega = \dot\Omega \cup \Gamma$ be a bounded $W^{2-1/r}_r$ domain. Let $\eta := \eta_+ \mathbb{1}_{\Omega_+} + \eta_- \mathbb{1}_{\Omega_-}$ for any $\eta_\pm > 0$, $\boldsymbol{u}_0 \in \widetilde{\mathcal{D}}^{2-2/p}_{q,p}(\dot\Omega)$ and $(\boldsymbol{f}, g, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{k}) \in \mathcal{Z}_{p,q,\varepsilon}$ for some $\varepsilon > 0$. Then (2.2) admits a unique solution $(\boldsymbol{u}, \mathfrak{q})$ with*

$$\boldsymbol{u} \in W^{2,1}_{q,p}(\dot\Omega\times\mathbb{R}_+) \ \text{ and } \ \mathfrak{q} \in L_p\big(\mathbb{R}_+; W^1_q(\dot\Omega) + \widehat{W}^1_{q,\Gamma_+}(\Omega)\big).$$

*Moreover, there exist constants $C$ and $\varepsilon_0$ $(\leq \varepsilon)$ such that*

$$\|e^{\varepsilon_0 t}(\partial_t\boldsymbol{u}, \boldsymbol{u}, \nabla\boldsymbol{u}, \nabla^2\boldsymbol{u})\|_{L_p(0,T; L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\mathfrak{q}\|_{L_p(0,T; W^1_q(\dot\Omega))} \leq C\Big(\|\boldsymbol{u}_0\|_{\widetilde{\mathcal{D}}^{2-2/p}_{q,p}(\dot\Omega)}$$
$$+ \|(\boldsymbol{f}, g, \boldsymbol{R}, \boldsymbol{h}, \boldsymbol{k})\|_{\mathcal{Z}_{p,q,\varepsilon_0}} + \delta(\Gamma_-)\sum_{\alpha=1}^M \Big(\int_0^T e^{p\varepsilon_0 t}|(\eta\boldsymbol{u}, \boldsymbol{p}_\alpha)_{\dot\Omega}|^p dt\Big)^{1/p}\Big).$$

*for any $T > 0$.*

## 3 Remark on the long time solvability in some fixed pool

In this part, we would like to give some simply application of Lemma 2.3 and Theorem 2.4 to the case of the bounded pool. That is, $\Omega := \dot\Omega \cup \Gamma$ is assumed to be some bounded $W^{2-1/r}_r$ domain with $\Gamma_+ = \emptyset$ hereafter. Assume that $(\boldsymbol{u}, \mathfrak{q})$ is a local-in-time solution of $(INS^{\mathfrak{L}}_{\pm})$ thanks to Theorem 2.1. Moreover, suppose that $T^\star$ is the lifespan of the solution $(\boldsymbol{u}, \mathfrak{q})$. By our discussions of Theorem 2.1, we have the continuity and non-degeneration of $\mathscr{A}_u\boldsymbol{u}$ across $\Gamma$. Thus we can reformulate the equations of $(\boldsymbol{u}, \mathfrak{q})$ as follows,

$$\begin{cases} \partial_t \boldsymbol{u} - \eta^{-1} \operatorname{Div}_\xi \mathbb{T}(\boldsymbol{u}, \mathfrak{q}) = \boldsymbol{f}_{u,\mathfrak{q}}, \quad \operatorname{div}_\xi \boldsymbol{u} = g_u = \operatorname{div}_\xi \boldsymbol{R}_u & \text{in} \quad \dot{\Omega}\times]0, T^\star[, \\ [\![\mathbb{T}(\boldsymbol{u}, \mathfrak{q})\boldsymbol{n}]\!] = [\![\boldsymbol{h}_{u,\mathfrak{q}}]\!], \quad [\![\boldsymbol{u}]\!] = \boldsymbol{0} & \text{on} \quad \Gamma\times]0, T^\star[, \\ \boldsymbol{u}_- = \boldsymbol{0} & \text{on} \quad \Gamma_-\times]0, T^\star[, \\ \boldsymbol{u}|_{t=0} = \boldsymbol{v}_0 & \text{in} \quad \dot{\Omega}, \end{cases} \tag{3.1}$$

where $(\boldsymbol{f}_{u,\mathfrak{q}}, g_u, \boldsymbol{R}_u, \boldsymbol{h}_{u,\mathfrak{q}}, \boldsymbol{k}_{u+\mathfrak{q}+})$ are defined by

$$\eta \boldsymbol{f}_{u,\mathfrak{q}} := -\operatorname{Div}_\xi \big(\mathbb{T}(\boldsymbol{u}, \mathfrak{q}) - \mathbb{T}_u(\boldsymbol{u}, \mathfrak{q})\mathscr{A}_u\big),$$

$$g_u := \nabla_\xi^\top \boldsymbol{u} : (\mathbb{I} - \mathscr{A}_u^\top), \quad \boldsymbol{R}_u := (\mathbb{I} - \mathscr{A}_u^\top)\boldsymbol{u},$$

$$\boldsymbol{h}_{u,\mathfrak{q}} := \mathbb{T}(\boldsymbol{u}, \mathfrak{q})\boldsymbol{n} - \mathbb{T}_u(\boldsymbol{u}, \mathfrak{q})\mathscr{A}_u\boldsymbol{n}.$$

Then the main task of this section is the following long time result concerning (3.1).

**Theorem 3.1.** *Let $(p, q) \in (I) \cup (II)$ as in Theorem 2.1 and $\Omega$ be a bounded $W_r^{2-1/r}$ $(r \geq q)$ pool with $\Gamma_+ = \emptyset$. Assume that $\rho_0(\xi) = \eta = \eta_+ \mathbb{1}_{\Omega_+} + \eta_- \mathbb{1}_{\Omega_-}$ and $\mu = \mu_+ \mathbb{1}_{\Omega_+} + \mu_- \mathbb{1}_{\Omega_-}$ are piecewise constant for any $\eta_\pm, \mu_\pm > 0$. If $\|\boldsymbol{v}_0\|_{\mathcal{D}_{q,p}^{2-2/p}(\dot{\Omega})} \ll 1$, then (3.1) admits a unique global solution $(\boldsymbol{u}, \mathfrak{q})$. Moreover, there exists constant $\varepsilon_0$ and $C$ such that*

$$\|e^{\varepsilon_0 t} \boldsymbol{u}\|_{W_{q,p}^{2,1}(\dot{\Omega}\times\mathbb{R}_+)} + \|e^{\varepsilon_0 t} \nabla \mathfrak{q}\|_{L_p(\mathbb{R}_+; L_q(\dot{\Omega}))} \leq C \|\boldsymbol{v}_0\|_{\mathcal{D}_{q,p}^{2-2/p}(\dot{\Omega})}.$$

**Remark.** *In fact, the assumption $\Gamma_+ = \emptyset$ in Theorem 3.1 does not matter in our framework. For simplicity, we here only focus on more interesting physical case without the surface $\Gamma_+$. This problem was also studied in [1] with imposing surface tension on the interface. The authors in [1] used so called* Hanzawa *transformation to fix the moving interface and then they established the solutions in $L_p - L_p$ maximal regularity class for $p > N + 2$. Thus Theorem 3.1 here can be regarded as a simple remark of the results in [1].*

In the rest of this part, we will outline the proof of Theorem 3.1 by applying the idea in [4]. It is convenient to use the notation

$$\mathcal{I}_{\varepsilon,\boldsymbol{v}}(a, b) := \|e^{\varepsilon t}(\partial_t \boldsymbol{v}, \boldsymbol{v}, \nabla \boldsymbol{v}, \nabla^2 \boldsymbol{v})\|_{L_p(a,b;L_q(\dot{\Omega}))},$$

for any vector $\boldsymbol{v}$, any time interval $]a, b[ \subset \mathbb{R}$ and any $\varepsilon > 0$.

**Step 1. Reduction**

As a starting point, we consider the following linear equations,

$$\begin{cases} \partial_t \boldsymbol{u}_{L} - \eta^{-1} \operatorname{Div}_\xi \mathbb{T}(\boldsymbol{u}_{L}, \mathfrak{q}_{L}) = \mathbf{0}, \ \operatorname{div}_\xi \boldsymbol{u}_{L} = 0 & \text{in} \quad \dot{\Omega} \times \mathbb{R}_+, \\ [\![\mathbb{T}(\boldsymbol{u}_{L}, \mathfrak{q}_{L}) \boldsymbol{n}]\!] = [\![\boldsymbol{u}_{L}]\!] = \mathbf{0} & \text{on} \quad \Gamma \times \mathbb{R}_+, \\ \boldsymbol{u}_{L,-} = \mathbf{0} & \text{on} \quad \Gamma_- \times \mathbb{R}_+, \\ \boldsymbol{u}_{L}|_{t=0} = \boldsymbol{v}_0 & \text{in} \quad \dot{\Omega}. \end{cases} \tag{3.2}$$

Then Theorem 2.4 yields that there exists some $\varepsilon_0 > 0$ such that

$$\mathcal{I}_{\varepsilon_0, \boldsymbol{u}_{L}}(0, \infty) + \| e^{\varepsilon_0 t} \mathfrak{q}_{L} \|_{L_p(\mathbb{R}_+; W^1_q(\dot{\Omega}))} \leq C \| \boldsymbol{v}_0 \|_{\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})} \ll 1. \tag{3.3}$$

Thus $(\boldsymbol{w}, P) := (\boldsymbol{u} - \boldsymbol{u}_{L}, \mathfrak{q} - \mathfrak{p}_{L})$ satisfies the following equations for any $0 < T < T^\star$,

$$\begin{cases} \partial_t \boldsymbol{w} - \eta^{-1} \operatorname{Div} \mathbb{T}(\boldsymbol{w}, P) = \boldsymbol{f}_{u,\mathfrak{q}}, \ \operatorname{div} \boldsymbol{w} = g_u = \operatorname{div} \boldsymbol{R}_u & \text{in} \quad \dot{\Omega} \times ]0, T], \\ [\![\mathbb{T}(\boldsymbol{w}, P) \boldsymbol{n}]\!] = [\![\boldsymbol{h}_{u,\mathfrak{q}}]\!], \ [\![\boldsymbol{w}]\!] = \mathbf{0} & \text{on} \quad \Gamma \times ]0, T], \\ \boldsymbol{w}_- = \mathbf{0} & \text{on} \quad \Gamma_- \times ]0, T], \\ \boldsymbol{w}|_{t=0} = \mathbf{0} & \text{in} \quad \dot{\Omega}. \end{cases} \tag{3.4}$$

**Step 2. Extension operators**

To apply our decay property, we need some extension operators. For any (scalar- or vector-valued) mapping $\mathfrak{h}$ defined on $]0, T]$ and any fixed parameter $t \in ]0, T]$, we denote that

$$E_{(t)} \mathfrak{h}(\cdot, s) := \begin{cases} \mathfrak{h}(\cdot, s) & \text{if} \quad s \in ]0, t[, \\ \mathfrak{h}(\cdot, 2t - s) & \text{if} \quad s \in ]t, 2t[, \\ 0 & \text{otherwise}. \end{cases}$$

Assume that $\varphi(s) \in C^\infty(\mathbb{R})$ is some cut-off function such that $\varphi(s) = 1$ for $s \leq 0$ and $\varphi(s) = 0$ for $s \geq 1$. Then denote $\varphi_t(s) := \varphi(s - t)$ for any $t \in ]0, T^\star[$. Now we introduce that the following operators

$$\begin{aligned} \eta \widetilde{\boldsymbol{f}}_{u,\mathfrak{q}} :=& - \operatorname{Div}_\xi \Big( \mu \big( \widetilde{\mathbb{H}}_u + \widetilde{\mathbb{D}}_u \big) \Big) + \operatorname{Div}_\xi \big( \mu \widetilde{\mathbb{H}}_u \cdot \varphi_{T}(t) E_{(T)} (\mathbb{I} - \mathscr{A}_u) \big) \\ &+ \operatorname{Div}_\xi \big( \widetilde{\mathfrak{q}} \cdot \varphi_{T}(t) E_{(T)} (\mathbb{I} - \mathscr{A}_u) \big), \\ \widetilde{\mathbb{H}}_u :=& \nabla_\xi^\top \widetilde{\boldsymbol{u}} \cdot \varphi_{T}(t) E_{(T)} (\mathbb{I} - \mathscr{A}_u^\top) + \varphi_{T}(t) E_{(T)} (\mathbb{I} - \mathscr{A}_u) \cdot \nabla_\xi \widetilde{\boldsymbol{u}}^\top, \\ \widetilde{\mathbb{D}}_u :=& \mathbb{D}(\widetilde{\boldsymbol{u}}) \cdot \varphi_{T}(t) E_{(T)} (\mathbb{I} - \mathscr{A}_u), \end{aligned}$$

$$
\begin{aligned}
\widetilde{g}_u &:= \nabla_\xi^\top \widetilde{\boldsymbol{u}} : \varphi_{T}(t) E_{(T)}(\mathbb{I} - \mathscr{A}_u^\top),\\
\widetilde{\boldsymbol{R}}_u &:= \varphi_{T}(t) E_{(T)}(\mathbb{I} - \mathscr{A}_u^\top)\widetilde{\boldsymbol{u}},\\
\widetilde{\boldsymbol{h}}_{u,\mathfrak{q}} &:= \mu\big(\widetilde{\mathbb{H}}_u + \widetilde{\mathbb{D}}_u\big)\boldsymbol{n} - \big(\mu\widetilde{\mathbb{H}}_u \cdot \varphi_{T}(t) E_{(T)}(\mathbb{I} - \mathscr{A}_u)\big)\boldsymbol{n}\\
&\quad - \big(\widetilde{\mathfrak{p}} \cdot \varphi_{T}(t) E_{(T)}(\mathbb{I} - \mathscr{A}_u)\big)\boldsymbol{n}.\\
\widetilde{\mathfrak{p}} &:= |\widetilde{\mathscr{A}_u}\boldsymbol{n}|^{-2}\mu\big(\widetilde{\mathbb{H}}_u + \mathbb{D}(\widetilde{\boldsymbol{u}})\big)\widetilde{\mathscr{A}_u}\boldsymbol{n} \cdot \widetilde{\mathscr{A}_u}\boldsymbol{n}\big)\\
\widetilde{\mathscr{A}_u} &:= \varphi_{T}(t)\big(E_{(T)}(\mathscr{A}_u - \mathbb{I}) + \mathbb{I}\big)
\end{aligned}
$$

It is not hard to observe that

$$
(\widetilde{\boldsymbol{f}}_{u,\mathfrak{q}}, \widetilde{g}_u, \widetilde{\boldsymbol{R}}_u, \widetilde{\boldsymbol{h}}_{u,\mathfrak{q}})|_{t\in]0,T]} = (\boldsymbol{f}_{u,\mathfrak{q}}, g_u, \boldsymbol{R}_u, \boldsymbol{h}_{u,\mathfrak{q}}) \text{ for any } 0 < T < T^\star.
$$

As we proved in [4], $(\widetilde{\boldsymbol{f}}_{u,\mathfrak{q}}, \widetilde{g}_u, \widetilde{\boldsymbol{R}}_u, \widetilde{\boldsymbol{h}}_{u,\mathfrak{q}}, \boldsymbol{0}) \in \mathcal{Z}_{p,q,\varepsilon_0}$ such that

$$
\|(\widetilde{\boldsymbol{f}}_{u,\mathfrak{q}}, \widetilde{g}_u, \widetilde{\boldsymbol{R}}_u, \widetilde{\boldsymbol{h}}_{u,\mathfrak{q}}, \boldsymbol{0})\|_{\mathcal{Z}_{p,q,\varepsilon_0}} \lesssim \big(\|\boldsymbol{v}_0\|^2_{\widetilde{\mathcal{D}}^{2-2/p}_{q,p}(\dot{\Omega})} + X(T)^2\big)\big(X(T)+1\big), \tag{3.5}
$$

with $X(T) := \mathcal{I}_{\varepsilon_0,\boldsymbol{w}}(0,T) + \|e^{\varepsilon_0 t} P\|_{L_p(0,T;W^1_q(\dot{\Omega}))}$.

**Step. 3 Construction of global-in-time solutions**

According to (3.4) we consider the following problem,

$$
\begin{cases}
\partial_t \boldsymbol{U} - \eta^{-1}\operatorname{Div}_\xi \mathbb{T}(\boldsymbol{U}, Q) = \widetilde{\boldsymbol{f}}_{u,\mathfrak{q}}, \ \operatorname{div}_\xi \boldsymbol{U} = \widetilde{g}_u = \operatorname{div}_\xi \widetilde{\boldsymbol{R}}_u & \text{in } \dot{\Omega}\times\mathbb{R}_+,\\
[\![\mathbb{T}(\boldsymbol{U}, Q)\boldsymbol{n}]\!] = [\![\widetilde{\boldsymbol{h}}_{u,\mathfrak{q}}]\!], \ [\![\boldsymbol{U}]\!] = \boldsymbol{0} & \text{on } \Gamma\times\mathbb{R}_+,\\
\boldsymbol{U}_- = \boldsymbol{0} & \text{on } \Gamma_-\times\mathbb{R}_+,\\
\boldsymbol{U}|_{t=0} = \boldsymbol{0} & \text{in } \dot{\Omega}.
\end{cases} \tag{3.6}
$$

Then apply Theorem 2.4 and (3.5) by noting the uniqueness of (3.1) on $]0,T]$,

$$
X(T) \lesssim \big(\|\boldsymbol{v}_0\|^2_{\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})} + X(T)^2\big)\big(X(T)+1\big),
$$

which, together with (3.3), gives us that

$$
X(T) \lesssim \|\boldsymbol{v}_0\|^2_{\mathcal{D}^{2-2/p}_{q,p}(\dot{\Omega})} + X(T)^2 + X(T)^3. \tag{3.7}
$$

Now recall the lemma below in [4].

**Lemma 3.2.** *Assume that $X(t) \geq 0$ is a continuous function on $[0,T] \subset [0,\infty[$ satisfying*

$$
X(t) \leq a + bX(t)^2 + bX(t)^3 \quad \forall\, t \in [0,T],
$$

*where $a, b > 0$ such that*

$$
a < r_b(2 - b\,r_b)/3, \quad X(0) \leq r_b, \quad r_b := (-1 + \sqrt{1 + 3b^{-1}})/3. \tag{3.8}
$$

*Then we have $X(t) \leq 2a$.*

Thus Lemma 3.2 and (3.7) yield that $X(T)$ is uniformly (with respect to $T$) bounded by some small constant $C\|\boldsymbol{v}_0\|^2_{\mathcal{D}^{2-2/p}_{q,p}(\dot\Omega)}$. Finally, our proof is complete by the standard bootstrap arguments.

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